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Tensors (continued). If there's time, we'll do groups too.

Tensor calculation rules.

1 addition, 2 outer multiplication, 3 contraction, 4 symmetrizations.

5. Quotient rule. (Important for showing that a given object is a tensor.)

Ex. Suppose you have $(1,0)$, C^μ, but you don't know if C is a tensor or not. Then if you know that the scalar product with $(0,1)$ tensor always gives a scalar, you conclude that C^μ is a tensor.

Proof:

$$C^\nu A_\nu = S$$

$$C'^\mu A'_\mu = S' = S$$

$$C'^\mu \frac{\partial x^\nu}{\partial x'^\mu} A_\nu = S$$

$$\left(C^\nu - C'^\mu \frac{\partial x^\nu}{\partial x'^\mu} \right) A_\nu = 0$$

$$\Rightarrow \quad C^\nu = C'^\nu \frac{\partial x^\nu}{\partial x'^\nu} \quad \Rightarrow \quad C^\nu \text{ is a tensor.}$$

6. Kronecker symbol.

$$\delta^\mu_\nu \equiv \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

can be regarded as a $(1,1)$ tensor field.

$$\delta'^\mu{}_\nu = \delta^\rho{}_\sigma \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} = \frac{\partial x'^\mu}{\partial x^\rho} \cdot \frac{\partial x^\rho}{\partial x'^\nu} = \delta^\mu{}_\nu$$

This is therefore called an *invariant tensor*.

7. M is called metric, or riemannian, manifold, if there is an invariant local distance

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

By the quotient rule $g_{\mu\nu}(x)$ is a $(0,2)$ tensor field and its inverse, $g^{\mu\nu}$, defined by $g^{\mu\rho}\,g_{\rho\nu} = \delta^\mu{}_\nu$. A $(2,0)$ tensor field. They are used as standard tensor for raising and lowering indices.

Ex. $A^\mu = g^{\mu\rho} A_\rho$, j^μ, $j_\mu \equiv g_{\mu\rho}\, j^\rho$.

Connection to F2 vector calculus course: There M was metric, dim = 3, coordinates assumed orthogonal. There one wrote instead of $\mathrm{d}x^\mu$

$$\mathrm{d}\boldsymbol{r} = \sum_\nu \mathrm{d}x^\nu\, h_\nu\, \hat{\boldsymbol{e}}_\nu = \sum_\nu \mathrm{d}x^\nu\, \boldsymbol{t}_\nu$$

$\hat{\boldsymbol{e}}_\nu$ was unit basis vector, h_ν was scale factor, $h_\nu \hat{\boldsymbol{e}}_\nu$ was tangent basis vector.

In general coordinates, are not orthogonal and one writes instead

$$\mathrm{d}x = \mathrm{d}x^\nu\, \partial_\nu = \sum_\nu \mathrm{d}x^\nu\, e_\nu{}^a\, \hat{e}_a$$

$\hat{e}_a$ orthonormal basis vectors, ∂_ν tangent basis vectors, $e_\nu{}^a(x)$ vielbeins.

$$\mathrm{d}s^2 = \mathrm{d}x \cdot \mathrm{d}x = \sum_{\mu\nu a b} \mathrm{d}x^\mu \, e_\mu{}^a \underbrace{\hat{e}_a \cdot \hat{e}_b}_{=\delta_{ab}} e_\nu{}^b \, \mathrm{d}x^\nu = \sum_{\mu\nu a} \mathrm{d}x^\mu \underbrace{e_\mu{}^a e_\nu{}^a}_{=g_{\mu\nu}(x)} \delta x^\nu = g_{\mu\nu} \mathrm{d}x^\mu \, \mathrm{d}x^\nu$$

8. Differentiation of a $(p,\ q)$ field object, usually does not produce a $(p,\ q\ +\ 1)$ tensor. The trouble is derivatives of the transformation matrix.

Example: $(0,1)$ tensor field $A_\mu(x)$.

$$A'_\nu(x') = \frac{\partial x^\sigma}{\partial x'^\nu} A_\sigma(x)$$

$$(\partial_\mu A_\nu)'(x') = \frac{\partial x^\rho}{\partial x'^\mu} \partial_\rho \left(\frac{\partial x^\sigma}{\partial x'^\nu} A_\sigma(x) \right) = \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} A_\sigma(x) + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \partial_\rho A_\sigma(x)$$

You have a tensor only if the first term always cancels. Important case: This happens for $\partial_\mu A_\nu - \partial_\nu A_\mu$. Since it is odd under transformation $\mu \leftrightarrow \nu$ and since the unwanted term is even. More generally, for the same reason, the totally antisymmetrized derivative of a totally antisymmetric $(0, q)$ tensor, is a $(0, q+1)$ tensor.

Remark: In gravity theory the problem is handled by introducing (1, 2) field $\Gamma_\mu{}^\nu{}_\rho$, not tensor, called Riemann connection, and covariant derivative

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma_\mu{}^\nu{}_\rho V^\rho$$

defined such that $D_\mu V^\nu$ is a tensor. Compare covariant derivative in Yang-Mills theory, $(A_\mu)^a{}_b$.

9. Tensor densities.

In gravity theory, action must be scalar.

$$A = \int \mathrm{d}^4 x \, \mathcal{L} = \int \mathrm{d}^4 x' \, \mathcal{L}'$$

$$\mathrm{d}^4 x' = \mathrm{d}^4 x \cdot \det\left(\frac{\partial x'}{\partial x} \right)$$

Determinant of transformation matrix $\frac{\partial x'^\rho}{\partial x^\sigma}$.

$$\Rightarrow \mathcal{L}' = \det\left(\frac{\partial x}{\partial x'} \right) \mathcal{L}$$

Requires that $\mathcal{L}$ is a scalar density of weight $w = 1$.

Example of scalar density: $\det g$, where g is the metric tensor $g_{\mu\nu}$.

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} g_{\rho\sigma} \frac{\partial x^\sigma}{\partial x'^\nu}$$

$$\det(g') = \left(\det\left(\frac{\partial x}{\partial x'} \right) \right)^2 \det(g)$$

$\det(g)$ is a scalar densito of weight $w = 2$.

$\sqrt{\det(g)}$ is a scalar density of weight $w = 1$.

In general one can work with objects which are (p, q) tensor densities of weight w.

Group theory

Definition of group: done.

Definition: A subset $H \subseteq G$ of group G is called subgroup of G if it is a group under the same composition law as G. $a, b \in H \Rightarrow a\, b \in H$, $e \in H$, $a \in H \Rightarrow a^{-1} \in H$.

Definition: A subgroup $H \subseteq G$ is called an invariant subgroup iff:

$$h \in H, g \in G \quad \Rightarrow \quad g\, h\, g^{-1} \in H$$

(h conjugated by g).

Definition: g_1, $g_2 \in G$ are said to be conjugate to each other iff $\exists g \in G$: $g_1 = g\, g_2\, g^{-1}$. Denoted $g_1 \sim g_2$.

Note: Being conjugate is an equivalence relation; for

1) Transitivity: $g_1 \sim g_2$, $g_2 \sim g_3 \Rightarrow g_1 \sim g_3$.

Proof: $g_1 = g\, g_2\, g^{-1} = g\big(g'\, g_3 g'^{-1}\big)\, g^{-1} = (g\, g')g_3\, (g\, g')^{-1}$

2) Symmetry: $g_1 \sim g_2 \Rightarrow g_2 \sim g_1$.

$$g_1 = g\, g_2\, g^{-1} \quad \Rightarrow \quad g_2 = \big(g^{-1}\big)\, g_1\, \big(g^{-1}\big)^{-1}$$

3) Relflexivity: $g \sim g$.

$$g = e\, g\, e^{-1}$$

It gives rise to a partitioning of G into union of disjunct sets, equivalence classes, conjugacy classes.

Example. SO(3) = group of all rotations (about origin) in $\mathbb{R}^3$. A $g \in \mathrm{SO}(3)$ is specified by rotation angle $0 \leqslant \theta \leqslant \pi$ and a direction $\hat{\boldsymbol{n}}$. Two rotations $R(\hat{\boldsymbol{n}}, \theta)$, $R(\hat{\boldsymbol{n}}, \theta')$ are conjugate if $\theta = \theta'$, for there is always another rotation R that rotates $\hat{\boldsymbol{n}}'$ to $\hat{\boldsymbol{n}}$. Then $R\, R(\hat{\boldsymbol{n}}', \theta')R^{-1} = R(\hat{\boldsymbol{n}}, \theta') = R(\hat{\boldsymbol{n}}, \theta)$ iff $\theta' = \theta$.

Definition: a group G is called commutative or Albelian if all its elements commute, i.e. g_1, $g_2 \in G \Rightarrow g_1\, g_2 = g_2\, g_1$. For albelian groups each elemet is its own conjugacy class.

Number of elements in a group can be finite or infinite. Among infinite groups there is a nice subclass, Lie groups, which are simultaneously smooth manifolds.