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Today Solitons.

Ex 1. O(1) model with spontaneous symmetry breaking.

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,\lambda\,(\phi^2 - c^2)^2, \quad c > 0$$

2 vacua $\phi_0 = c$, $\phi_0 = -c$.

Now chose number of space time dimeensions $d = 1 + 1$.

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2\right) - \frac{1}{2}\,\lambda\,(\phi^2 - c^2)^2$$

what happens if $x \to +\infty$ and $x \to -\infty$ choose different vacua.

$$\phi_s(x) \to +c, \quad x \to +\infty$$

$$\phi_s(x) \to -c, \quad x \to -\infty$$

Then $\phi_s(x)$ is a static solution to equations minimizing potential energy U.

$$U = \int \mathrm{d}x\left(\frac{1}{2}\,\phi'^2 + \frac{1}{2}\,\lambda\,(\phi^2 - c^2)^2\right) = \int \mathrm{d}x\,\mathcal{L}(\phi, \phi')$$

Equation: $\partial_x^2\phi_s = \lambda\,\phi(\phi^2 - c^2)$.

No explicit x dependency in $U \Rightarrow$ Jacobi integral conserved.

$$J = \frac{\mathrm{d}}{\mathrm{d}x}\,\frac{\partial L}{\partial\phi'} - \frac{\partial L}{\partial\phi} = \frac{1}{2}\,\phi'^2 - \frac{1}{2}\,\lambda\,(\phi^2 - c^2)^2 = \text{constant} = 0$$

$$\phi' = \pm\sqrt{\lambda}(c^2 - \phi^2)$$

$$\frac{\phi'}{c} = \sqrt{\lambda}\,c\left(1 - \frac{\phi^2}{c^2}\right)$$

$$\frac{\phi}{c} = \tanh\left(\sqrt{\lambda}\,c\,(x - x_0)\right)$$

This configuration is called soliton, energy localized to region of width $\Delta x = 1/\sqrt{\lambda}\,c$ around $x = x_0$. Total energy = soliton mass $\sim\sqrt{\lambda}\,c^3$.

$\phi_s(x - x_0)$ describes soliton (extended particle) at rest at x_0. This soliton at rest, but the theory is relativistic $\Rightarrow$ by Lorentz transformation you get moving solution with constant velocity v and relativistic energy momentum relation $E^2 = m^2 + p^2$, $v = p/E$. This type of soliton could occure as a domain wall in astrophysics.

Solitons in $d > 1 + 1$?

O(n) model in $d - 1$ space dimensions.

$n = 2$. O(2) model in $d = 2 + 1$, 2 space dimensions. In vacuum $\boldsymbol{\phi}_0 = c\,\hat{\boldsymbol{n}}$, ($\hat{\boldsymbol{n}}$ unit vector in x, y space). Example, n points radially $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}}$ in polar coordinates (ρ, α). Internal space ϕ_1, ϕ_2. For $|\boldsymbol{r}| >$ soliton radius $\phi_s(x)$ is a map from $R^{d-1} - \{|\boldsymbol{r}| <$ soliton radius$\}$ to manifold of possible vacua.

Ansatz for soliton in U(1) notation:

$$\phi(\rho,\alpha)=f(\rho)\,\mathrm{e}^{\mathrm{i}\alpha}$$

$$U=\int \mathrm{d}^2x\,\frac{1}{2}(\nabla\phi)^*\cdot(\nabla\phi)+\frac{1}{4}\lambda(\phi^*\phi-c^2)^2$$

$$=\int \mathrm{d}\rho\,2\pi\rho\left(\frac{1}{2}(f')^2+\frac{1}{2\,\rho^2}+\frac{1}{4}\lambda\,(f^2-c^2)^2\right)$$

Unfortunately this does not work, for

$$\int \mathrm{d}\rho\,\rho\,\frac{1}{\rho^2}=\infty$$

Diverges at infinity $\Rightarrow$infinite energy.

Gauged O(2) model in $d=2+1$.

$$\mathcal{L}=(D_\mu\phi)^*(D^\mu\phi)-\frac{1}{4}\lambda\,(\phi^*\phi-c^2)^2-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu\phi=(\partial_\mu-\mathrm{i}\,A_\mu)$$

Ansatz: $\phi_s(\rho,\alpha)=c\,\mathrm{e}^{\mathrm{i}\alpha}\chi(\rho)$. $\chi(0)=0$, $\chi(\infty)=1$.

$$A_0=0,\quad \boldsymbol{A}(\rho,\alpha)=\hat{\boldsymbol{\alpha}}\,A_\alpha(\rho)$$

$$\boldsymbol{D}\phi=(\nabla-\mathrm{i}\boldsymbol{A})\phi=\left(\hat{\boldsymbol{\rho}}\,\frac{\partial}{\partial\rho}+\frac{1}{\rho}\hat{\boldsymbol{\alpha}}\,\frac{\partial}{\partial\alpha}-\mathrm{i}\,\boldsymbol{A}\right)\phi=c\,\mathrm{e}^{\mathrm{i}\alpha}\left[\hat{\boldsymbol{\rho}}\,\chi'+\mathrm{i}\,\hat{\boldsymbol{\alpha}}\left(\frac{1}{\rho}-A_\alpha\right)\right]\chi$$

$$\nabla\times\boldsymbol{A}=\frac{1}{\rho}\,\partial_\rho(\rho\,A_\alpha)$$

$$U=\int \mathrm{d}^2x\left[(\boldsymbol{D}\phi)^*\cdot(\boldsymbol{D}\phi)+\frac{1}{4}\lambda\,(\phi^*\phi-c^2)^2+\frac{1}{2\,g^2}(\nabla\times\boldsymbol{A})^2\right]$$

$$=2\pi\,c^2\int_0^\infty \mathrm{d}\rho\,\rho\left[\chi'^2+\left(\frac{1}{\rho}-A_\alpha\right)^2\chi^2+\frac{1}{4}\lambda\,c^2(1-\chi^2)^2+\frac{1}{2\,g^2}\,\frac{1}{\rho^2}(\partial_\rho(\rho\,A_\alpha))^2\right]$$

It is easy to see that minimum exists and $0<U<\infty$, so this solution exists. In Landau Ginsburg model it describes flux tube in superconductor. Magnetic flux through tube:

$$\int_0^{2\pi} A_{\mathrm{em}}\,\rho\,\mathrm{d}\alpha=\frac{1}{2\,e}\int A\,\rho\,\mathrm{d}\alpha=\frac{2\pi}{2e}$$

$g\to 2e$, $A_\mu\to 2\,e\,A_{\mathrm{em}}$.

Solitons in $d=3+1$? Yes.

Gauged O(3) model in $3+1$ dimensions with spontaneous symmetry breaking has solitions too.

First spontaneous symmetry breaking in O(3) model.

In ungauged model O(3) was spontaneously broken to O(2)and there are 2 Goldstone bosons. In gauged model this means that 2 of the 3 vector fields become massive, one remains massless, called EM field, one can make ansatz and find stable soliton as in O(2) case. This soliton has magnetic charge $q=4\pi/e$.