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Example of group: The set of all symmetry transformations which leave the object invariant. Composition is just two transformations combined, $O_1 \circ O_2 \varphi = (O_1(O_2\varphi))$. Identity = "no transformation at all", inverse = reverse transformation. In fact, group is a mathematical abstraction of symmetry transformations. Groups are important in physics because symmetry is important and group theory is the way to study symmetries.

$\mathrm{U}(n)$ and $\mathrm{O}(n)$ are two examples of continuous groups, i.e. the elements are enumerated by continuous parameters. (These groups are smooth manifolds.) In general they are different, but there are exceptions: $\mathrm{U}(1) = \mathrm{SO}(2)$, $\mathrm{SU}(2)/Z_2 = \mathrm{SO}(3)$. S stands for special, means determinant $= 1$. $Z_2 = \{1, -1\}$ as a group.

First case:

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = O\varphi = \begin{pmatrix} \cos\theta\,\varphi_1 - \sin\varphi\,\varphi_2 \\ \sin\theta\,\varphi_1 + \cos\theta\,\varphi_2 \end{pmatrix}$$

Put $\varphi_1 + \mathrm{i}\,\varphi_2 = \sqrt{2}\,\psi \Rightarrow$

$$\sqrt{2}\,\psi' = \varphi_1' + \mathrm{i}\,\varphi_2' = (\cos\theta + i\sin\theta)\,\varphi_1 + (-\sin\theta + \mathrm{i}\cos\theta)\varphi_2 =$$

$$= \mathrm{e}^{\mathrm{i}\theta}(\varphi_1 + \mathrm{i}\,\varphi_2) = \mathrm{e}^{\mathrm{i}\theta}\sqrt{2}\,\psi$$

Shows $\mathrm{U}(1) = \mathrm{SO}(2)$.

Electrodynamics of charged scalar field in $\mathrm{U}(1)$ notation.

$$\mathcal{L} = \frac{1}{4\,g^2} G_{\mu\nu} G^{\mu\nu} + (D_\mu\psi)^*\, D^\mu\psi - m^2\,\psi^*\psi$$

with $D_\mu\psi = (\partial_\mu - \mathrm{i}\,A_\mu)\psi$. Gauge transformation: $\psi \to \mathrm{e}^{\mathrm{i}\theta(x)}\psi$, $A_\mu \to A_\mu + \partial_\mu\theta(x)$.

Remark: The standard model in particle physics, which describes all of physics except gravity, is described by a Lagrangian built as follows:

1) Choose matter fields, a set of free fields describing three generations of quarks and leptons.

2) Give them masses, using a scalar field and the Higgs mechanism, see below.

3) Choose gauge symmetry group, gauge it. That is all.

Goldstone's theorem and the Higgs mechanism.

Example: Our standard $\mathrm{O}(n)$ scalar field theory, ungauged, but with a new potential.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} - \underbrace{\frac{1}{4}\lambda\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi} - c^2)^2}_{=V(\boldsymbol{\phi})} = \frac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} - \frac{1}{4}\lambda\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 + \frac{1}{2}\lambda\,c^2\,\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{1}{4}\lambda\,c^4$$

Standard procedure to analyse the field theory.

1) Find the vaccuum state $\boldsymbol{\phi} = \boldsymbol{\phi}_0 =$ lowest energy state.

2) $\boldsymbol{\phi}(x,t) = \boldsymbol{\phi}_0 + \boldsymbol{\varphi}(x,t)$ with $\boldsymbol{\varphi}$ assumed small $\Rightarrow \mathcal{O}(\boldsymbol{\varphi}^2)$-terms describe particle content.

3) Terms of order $\boldsymbol{\varphi}^3$ and heigher describe interactions.

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1. V bounded from below $\Rightarrow \lambda > 0$. Two cases: a) $c^2 \leqslant 0$, $\boldsymbol{\phi}_0 = 0$ b) $c^2 > 0$, $|\boldsymbol{\phi}_0| = c > 0$.

2. a) $\partial_\mu \partial^\mu \boldsymbol{\varphi} - \lambda\, c^2\, \boldsymbol{\varphi} = 0$. n scalar fields, of mass $m^2 = \lambda\, c^2$.

   b) $|\boldsymbol{\phi}_0| = c$, $\partial_\mu \boldsymbol{\phi}_0 = 0$ but $\hat{\boldsymbol{\phi}}$ undetermined. Number of possible vacua $= \infty$. S^{n-1} manifold of pssible vacua.

$$\frac{1}{4}\,\lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi} - c^2) = \frac{1}{4}\,\lambda\,((\boldsymbol{\phi}_0 + \boldsymbol{\varphi})^2 - c^2)^2 = \frac{1}{4}\,\lambda(2\,\boldsymbol{\phi}_0\cdot\boldsymbol{\varphi} - (\boldsymbol{\varphi}^2))^2 = \lambda\,(\boldsymbol{\phi}_0\cdot\boldsymbol{\varphi})^2 + \mathcal{O}(\boldsymbol{\varphi}^3)$$

   The component of $\boldsymbol{\varphi} \| \boldsymbol{\phi}_0$ has mass $m = 2\,\lambda\, c^2$. Components of $\boldsymbol{\varphi} \perp \boldsymbol{\phi}_0$ are massless.

   Also, the vaccuum state $\boldsymbol{\phi}_0$ invariant only under $\mathrm{O}(n-1)$, One says, $\mathrm{O}(n)$ symmetry spontaneously broken to $\mathrm{O}(n-1)$.

This example illustrates Glodstone's theorem: when global symmetry group G (here $\mathrm{SO}(n)$) is broken spontaneously to $H \subset G$ (here $H = \mathrm{SO}(n-1)$) then there appears as many massles fields as there are broken generators.

$$\dim G - \dim H \ \text{ here: } \ \dim \mathrm{O}(n) - \dim \mathrm{O}(n-1) = \frac{1}{2}\, n\,(n-1) - \frac{1}{2}(n-1)(n-2) = n-1$$

dim $\mathrm{O}(n)$ is $\frac{1}{2}n\ (n-1)$ because the infinitesimal orthogonal transformation is described by an antisymmetric $n \times n$ matrix, having $n\,(n-1)/2$ entries.

$$O = 1 + \varepsilon\, A, \quad O\, O^T = 1, \quad (1 + \varepsilon\, A)(1 + \varepsilon\, A^T) = 1$$

$$\varepsilon(A + A^T) + \mathcal{O}(\varepsilon^2) = 0, \quad A + A^T = 0$$

"Explanation" Broken generators transform vacuum states. Fluctuations in such directions cost no energy.

O(2) case more eplicitly in U(1) notation:

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \frac{1}{2}\,\lambda\,(\phi^* \phi - c^2)^2$$

Case $\lambda > 0, c > 0$. Choose $\phi_0 = c$.

$$\phi(x,t) = \phi_0 + \varphi(x,t)$$

can be alternatively written as

$$\phi(x,t) = (c + \chi(x,t))\, \mathrm{e}^{\mathrm{i}\theta(x,t)}, \quad \theta, \chi \text{ real fields}$$

$$\partial_\mu \phi = ((c + \chi)\,\mathrm{i}\,\partial_\mu \theta + \partial_\mu \chi)\, \mathrm{e}^{\mathrm{i}\theta(x,t)}$$

$$\mathcal{L} = \chi_{,\mu}\chi^{,\mu} + (c + \chi)^2 \theta_{,\mu}\theta^{,\mu} - \frac{1}{2}\lambda\,((c + \chi)^2 - c^2)^2$$

$$m_\theta^2 = 0, \quad m_\chi^2 = 2\,\lambda\, c^2$$

SSB in gauged model.

Ex U(1), i.e. ED with interacting charged scalar field:

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - \frac{1}{2}\,\lambda\,(\phi^* \phi - c^2)^2 - \frac{1}{4\,g^2}\, F_{\mu\nu}\, F^{\mu\nu}$$

$\mathcal{L}$ invariant under gauge transformations $\phi = \mathrm{e}^{\mathrm{i}\Lambda(x)}\phi$.

Case $\lambda > 0, c > 0$. $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$

$$\Rightarrow |\phi_0| = c$$

Now phase of ϕ_0 can be made $= 0$ by gauge transformation. Choose $\phi_0 = c$.

$$\phi(x,t) = (c + \chi(x,t))\, \mathrm{e}^{\mathrm{i}\theta(x,t)}$$

$$\Rightarrow \partial_\mu \phi = \mathrm{e}^{\mathrm{i}\theta}(\partial_\mu \chi + (c+\chi)\, \mathrm{i}\, (\partial_\mu \theta - A_\mu))$$

$$\mathcal{L} = \partial_\mu \chi\, \partial^\mu \chi + (c+\chi)^2 (A_\mu - \partial_\mu \theta)(A^\mu - \partial^\mu \theta) - \frac{1}{2}\, \lambda ((c+\chi)^2 - c^2)^2 - \frac{1}{4\, g^2} F_{\mu\nu} F^{\mu\nu}$$

The field θ disappears by redefinition of the gauge field $A_\mu \rightarrow A_\mu + \partial_\mu \theta$.

Result: only one scalar field, χ, $m_\chi^2 = 2\, \lambda\, c^2$ and massive vector fields.

$$m_A^2 = 2\, g^2\, c^2$$

Illustrates Higg's mechanism: When local symmetry is broken spontaneously there is no Goldstone boson appears. Instead the gauge field gets mass. One says: "The gauge field eats the Goldstone boson and becomes massive". ["That's a funny way to express what is happening", Per remarks dryly.]

Note: the number of degrees of freedom is preserved. In our example before spontaneous symmetry breaking scalars 2 field degrees of freedom, vector 2 field degreees of freedom; after scalars 1, vector massive 3. In each case it adds up to four.