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Hamiltonian for the electromagnetic field

$$A = \int \mathrm{d}^4 x \, \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu = \frac{1}{2}(\dot{\boldsymbol{A}} + \nabla\varphi)^2 - \frac{1}{2}(\nabla \times \boldsymbol{A})^2 - \varphi\,\rho + \boldsymbol{A}\cdot\boldsymbol{j}$$

j^μ must be conserved. $\partial_\mu j^\mu = 0$.

Under gauge transformation $A_\mu \to A_\mu + \delta A_\mu$, $\delta A_\mu = \partial_\mu \Lambda(x,t)$. $\delta A =$ boundary term $\Rightarrow$ equations invariant. $\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0$.

$$\delta A_\mu j^\mu = (\partial_\mu \Lambda)\, j^\mu = \partial_\mu(\Lambda\, j^\mu) - \Lambda\, \partial_\mu j^\mu$$

The first is a boundary term in the action, the second is zero since j^μ is conserved.

From notes by Ingmar Bengtsson: "constrained Hamiltonian systems", linked from home page.

How to count the number of degrees of freedom of field?

Recipe: Put the system in a box, Fourier decompose the field.

$$\varphi(x,t) = \sum_k \tilde{\varphi}_k(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$A = \frac{1}{2}\int \mathrm{d}^4 x\, \partial_\mu\varphi\, \partial^\mu\varphi \propto \sum_k \dot{\tilde{\varphi}}_{-k}\dot{\tilde{\varphi}}_k - \boldsymbol{k}^2\, \tilde{\varphi}_{-k}\tilde{\varphi}_k$$

Definition: Number of field degrees of freedom of a field = number of degrees of freedom per Fourier component. I.e., a real scalar field has the number of degrees of freedom = 1.

A_μ has four real components, but A_4 has no time derivative in the action $\Rightarrow$number of degrees of freedom $\leqslant 3$. I will now apply Dirac's method for analysing systems with constraints. Its further development leads to BRST-quantisation method.

To make the example more instructive, I add a mass term:

$$\mathcal{L} = \frac{1}{2}\,(\dot{\boldsymbol{A}} + \nabla\varphi)^2 - \frac{1}{2}(\nabla\times\boldsymbol{A})^2 - \frac{1}{2}\,m^2\,(\boldsymbol{A}^2 - \varphi^2) - \rho\varphi + \boldsymbol{j}\cdot\boldsymbol{A}$$

Note that the mass term destroys gauge invariance:

$$\delta\frac{1}{2} m\, A_\mu\, A^\mu = m\, A^\mu\, \partial_\mu \Lambda \neq \text{total time derivative.}$$

Conjugate momenta:

$$\boldsymbol{\Pi} = \frac{\partial L}{\partial \dot{\boldsymbol{A}}} = \dot{\boldsymbol{A}} + \nabla\varphi, \quad \Pi_0 = \frac{\partial L}{\partial\dot{\varphi}} = 0 \quad \text{(primary constraint)}$$

Schematically:

$$H(p,q,\dot{q}) = \sum_\nu p_\nu \dot{q}_\nu - L$$

$$\delta H = \dot{q}^\nu \delta p_\nu + \underbrace{p_\nu \delta\dot{q}^\nu - \frac{\partial L}{\partial \dot{q}^\nu}\,\delta\dot{q}^\nu}_{=0} - \frac{\partial L}{\partial q^\nu}\,\delta q^\nu = \dot{q}^\nu \delta p_\nu - \dot{p}_\nu\, \delta q^\nu$$

⇒Hamilton's equations if δp and δq are arbitrary. Our constraint ($\Pi_0 = 0$) forbids independent variations. Remedy: add Lagrange multiplier term.

$$\mathcal{H} = \boldsymbol{\Pi} \cdot \dot{\boldsymbol{A}} - \mathcal{L} + u\,\Pi_0 = \frac{1}{2}\boldsymbol{\Pi}^2 - \boldsymbol{\Pi} \cdot \nabla\varphi + u\,\Pi_0 + \frac{1}{2}(\nabla \times \boldsymbol{A})^2 + \frac{1}{2}m^2\,(\boldsymbol{A}^2 - \varphi^2) + \rho\,\varphi - \boldsymbol{j} \cdot \boldsymbol{A}$$

$$\rightarrow \quad \frac{1}{2}\boldsymbol{\Pi}^2 + \varphi\,(\rho + \nabla \cdot \boldsymbol{\Pi}) + u\,\Pi_0 + \frac{1}{2}(\nabla \times \boldsymbol{A})^2 + \frac{1}{2}m^2\,(\boldsymbol{A}^2 - \varphi^2) - \boldsymbol{j} \cdot \boldsymbol{A}$$

Now one can write Hamilton's equations. We must check that the constraint stays zero.

$\Pi_0 = 0$,

$$\dot{\Pi}_0 = [\Pi_0, H]_{\mathrm{PB}} = -\frac{\partial H}{\partial \varphi} = m^2\varphi - (\rho + \nabla \cdot \boldsymbol{\Pi}) = 0$$

Secondary constraint.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho + \nabla \cdot \boldsymbol{\Pi} - m^2\varphi) = \dot{\rho} - \nabla \cdot \frac{\partial H}{\partial \boldsymbol{A}} - m^2\,\frac{\partial H}{\partial \Pi_0} = \underbrace{\dot{\rho} + \nabla \cdot \boldsymbol{j}}_{=0} - m^2\,\nabla \cdot \boldsymbol{A} - m^2 u = 0$$

No more constraints. Now there are two cases.

1) $m^2 \neq 0$ then u is fixed. $u = -\nabla \cdot \boldsymbol{A}$.

Sum up: 2 constraints

$$\begin{cases} \Pi_0 = 0 \\ \rho + \nabla \cdot \Pi - m^2\varphi = 0 \end{cases}$$

$$[(\rho + \nabla \cdot \Pi - m^2\varphi)(x), \Pi_0(x')]_{\mathrm{PB}} = -m^2\,\delta^3(\boldsymbol{x} - \boldsymbol{x}') \neq 0$$

Constraints are called *second class*. Treatment: Replace Poisson bracket with Dirac bracket. The Dirac bracket is a Poisson backet modified so that constraints have zero Dirac bracket with everything. In the present case this means Π_0 and φ are eliminated everywhere using constraints. After that, brackets of $\boldsymbol{\Pi}$ and $\boldsymbol{A}$ are as usual, and of Π_0, φ zero. Result:

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\Pi}^2 + \frac{1}{2}(\nabla \times \boldsymbol{A})^2 + \frac{1}{2m}(\rho + \nabla \cdot \boldsymbol{\Pi})^2 + \frac{1}{2}m^2\,\boldsymbol{A}^2 - \boldsymbol{j} \cdot \boldsymbol{A}$$

Finished. Conclusion: the number of degrees of freedom = 3.

2) $m^2 = 0$. Constraints $\Pi_0 = 0$, $\rho + \nabla \cdot \boldsymbol{\Pi} = 0$.

$$[\rho + \nabla \cdot \boldsymbol{\Pi}, \Pi_0]_{\mathrm{PB}} = 0$$

Constraints called first class.

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\Pi}^2 + \frac{1}{2}(\nabla \times \boldsymbol{A})^2 - \boldsymbol{j} \cdot \boldsymbol{A} + \varphi(\nabla \cdot \boldsymbol{\Pi} + \rho) + u\,\Pi_0$$

$$\dot{\varphi} = [\varphi, H]_{\mathrm{PB}} = u$$

⇒interpretation of u. But u is arbitrary. $\dot{\varphi} = u$ arbitrary ⇒φ arbitrary ⇒φ nonphysical ⇒φ, Π_0 can be dropped.

$$\Rightarrow \mathcal{H} = \frac{1}{2}\boldsymbol{\Pi}^2 + \frac{1}{2}(\nabla \times \boldsymbol{A}) - \boldsymbol{j} \cdot \boldsymbol{A}$$

One constraint remains: $\rho + \nabla \cdot \mathbf{\Pi} = 0$ (Gauss law: $\nabla \cdot \boldsymbol{E} + \rho = 0$) which generates gauge transformations.

$$\left[\boldsymbol{A}(x), \int \mathrm{d}^3x \, \Lambda(x) \nabla \cdot \mathbf{\Pi} + \rho\right]_{\mathrm{PB}} = -\nabla \Lambda(x)$$

So we have six field phase space dimensions. Variables $\boldsymbol{A}$, $\mathbf{\Pi}$. But points in it correspond to a physcial state only if they lie on a five field phase space dimensional submanifold $\rho + \nabla \cdot \mathbf{\Pi} = 0$. Morover points on this five-dimensional manifold correspond to the same physical state if they are connected by a gauge transformation.

Phase space of different physical states has field phase space dimension $6 - 1 - 1 = 4$. The electromagnetic field has four degrees of freedom.