2008–09–26

Mechanics of fields. Field theories are important in physics. All analytic mechanics methods developed so far are equally valid for field theories. It is just a matter of knowing a translation dictionary. The correspondence is:

Particle theory	Field theory
$q^\nu(t)$	$\varphi(x,t)$
ν	x
$v^\nu = \dot{q}^\nu$	$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} = \dot{\varphi}$
$p \sim m\dot{q}^\nu$	$\pi(x,t) \sim \text{const.}\,\dot{\varphi}$
$[q^\mu, \pi_\nu]_{\mathrm{PB}} = \delta^\mu_\nu$	$[\varphi(x,t), \pi(x,t)]_{\mathrm{PB}} = \delta^3(x-y)$
$\sum_\nu$	$\int \mathrm{d}^3x$

Example: Longitudinal sound waves in a homogeneous elastic rod. $\varphi(x,\, t)$ is the displacement, i.e. point of rod, which in equilibrium sists at x, now att ime t is at $x + \varphi(x,\, t)$. We only consider small displacements, and impose $|\mathrm{d}\varphi/\mathrm{d}x| \ll 1$.

$$T = \frac{1}{2}\int_0^l \mu\left(\frac{\partial\varphi}{\partial t}\right)^2 \mathrm{d}x, \quad \text{where } \mu \text{ is mass per unit length.}$$

$$V = \frac{1}{2}\int_0^l \mathrm{d}x\, \kappa\left(\frac{\partial\varphi}{\partial x}\right)^2$$

$$\frac{\partial\varphi}{\partial x} = \text{stretching}, \quad \kappa\,\frac{\partial\varphi}{\partial x} = \text{tension.}$$

$$A = \int \mathrm{d}t (T - V) = \frac{1}{2}\int_0^l \mathrm{d}x \int \mathrm{d}t\left(\mu\left(\frac{\partial\varphi}{\partial t}\right)^2 - \kappa\left(\frac{\partial\varphi}{\partial x}\right)^2\right) = \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}x\left(\mu(\partial_t\varphi)^2 - \kappa(\partial_x\varphi)^2\right)$$

Equations obtained by varying φ. $\varphi \to \varphi + \delta\varphi$.

$$\delta A = \int_{t_1}^{t_2}\int_0^l \mathrm{d}x(\mu\,\partial_t\varphi\,\partial_t\delta\varphi - \kappa\partial_x\varphi\partial_x\delta\varphi) =$$

$$= \int_{t_1}^{t_2}\int_0^l \mathrm{d}x\big(-\mu\,\partial_t^2\varphi + \kappa\,\partial_x^2\varphi\big)\delta\varphi + \int \mathrm{d}x\,\mu\,\partial_t\varphi\,\delta\varphi\Big|_{t_1}^{t_2} - \int \mathrm{d}t\,\kappa\,\partial_x\varphi\,\delta\varphi\Big|_{x=0}^{l}$$

$= 0$ for arbitrary allowed variations $\delta\varphi(x,t)$. Required: $\delta\varphi(x,t_1) = 0 = \delta\varphi(x,t_2)$.

Gives:

1. Equation of motions:

$$\mu\,\partial_t^2\varphi - \kappa\,\partial_x^2\varphi = 0, \quad 0 \leqslant x \leqslant l$$

$\Rightarrow$ velocity of sound $v = \sqrt{\kappa/\mu}$.

2. Boundary condition:

$$\partial_x\varphi\,\delta\varphi\big|_{x=0} = 0, \quad \partial_x\varphi\,\delta\varphi\big|_{x=l} = 0$$

2a: rod fastened at $x = 0, x = l$, then $\delta\varphi = 0$ at $x = 0, l$.

2b: rod ends free: then $\partial_x\varphi = 0$ at $x = 0, l$.

A bit more generally:

$$A = \int \mathrm{d}t \int \mathrm{d}^3x\, \mathcal{L}(\varphi, \partial_\mu\varphi) \equiv \int \mathrm{d}t\, L(\varphi, \partial_\mu\varphi)$$

$$\partial_\mu\, \varphi = (\partial_0\varphi, \partial_1\varphi, \partial_2\varphi, \partial_3\varphi) = (\partial_t\varphi, \nabla\varphi)$$

Equations:

$$\partial_\mu \frac{\partial\mathcal{L}(\varphi, \partial\varphi)}{\partial\varphi_{,\mu}} - \frac{\partial\mathcal{L}(\varphi, \partial\varphi)}{\partial\varphi} = 0$$

Important area of application: realtivistic physics.

Example: Scalar field $\varphi(t,\, \boldsymbol{x})$ in relativistic theory. Action built from scalar quantities: φ, φ^n, $\eta^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi$. Free scalar field of mass m:

$$A = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L} = \int \mathrm{d}^4x\, \left(\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right)$$

Equations:

$$\left(\nu^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\varphi = 0, \quad \left(\partial_t^2 - \nabla^2 + m^2\right)\varphi = 0$$

A wave equation. Linear $\rightarrow$ easy to solve. Ansatz:

$$\varphi(x, t) = \tilde{\varphi}(\boldsymbol{k}, t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

Equation of motion:

$$(\partial_t^2 + k^2 + m^2)\tilde{\varphi}(\boldsymbol{k}, t) = 0$$

$$\ddot{\tilde{\varphi}} + \omega_k^2\tilde{\varphi} = 0$$

Harmonic oscillator equation. General solution:

$$\varphi(x, t) = \int \mathrm{Re}\, \mathrm{d}^3k\, \tilde{\varphi}(k, t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$$

There is a procedure, quantisation, to make a quantum theory of this. It descrbies free particles of mass $M = \hbar\, m$, momentum $p = \hbar k$ and energy $E = \hbar\omega$ and relativistic energy-momentum relation $E^2 = M^2 + \boldsymbol{p}^2$ from $\omega^2 = m^2 + k^2$.

Hamiltonian formulation

Use the same $\mathcal{L}$ as before.

$$L = \int \mathrm{d}^3x\, \frac{1}{2}\, (\partial_t\varphi)^2 + \cdots$$

$$\delta L = \int \mathrm{d}^3x (\partial_t\varphi\, \delta_t\varphi + \cdots) = \int \mathrm{d}^3x\, \Pi(x, t)\, \delta\partial_t\varphi$$

$$\Pi(x, t) = \frac{\partial\mathcal{L}(x, t)}{\partial\varphi_{,0}(t, x)} = \frac{\delta L(t)}{\delta\varphi_{,t}(t, x)}$$

$$H(\varphi, \nabla, \Pi) = \int \mathrm{d}^3x (\Pi(x, t)\dot{\varphi}(x, t) - \mathcal{L})|_{\dot{\varphi} = H} = \int \mathrm{d}^3x\, \mathcal{H}$$

Equations:

$$\dot{\varphi}(x,t)=\frac{\delta H}{\delta \Pi}=\frac{\partial \mathcal{H}}{\partial \pi(x,t)}=\Pi(x,t)$$

$$\dot{\Pi}(x,t)=-\frac{\delta H}{\delta \varphi(x,t)}=-\left(-\nabla^2+m^2\right)\varphi(x,t)$$

$$\frac{\delta}{\delta\varphi(x)}\int (\nabla\varphi)^2 \mathrm{d}^3x=-\nabla^2\varphi(x)$$

$$\delta\int (\nabla\varphi)^2\mathrm{d}^3x=2\int \nabla\varphi\,\delta\,\nabla\varphi\,\mathrm{d}^3x=-2\int \left(-\nabla^2\varphi\right)\delta\varphi\,\mathrm{d}^3x$$

Poisson bracket:

$$[A,B]_{\mathrm{PB}}=\int \mathrm{d}^3x\left(\frac{\delta A}{\delta\varphi(x,t)}\frac{\delta B}{\delta\pi(x,t)}-\frac{\delta A}{\delta\pi(x,t)}\frac{\delta B}{\delta\varphi(x,t)}\right)$$

A and B are functions of t.

$$\dot{A}(t)=[A,H]$$

$$\mathcal{H}=\frac{1}{2}\left(\Pi^2(x,t)+(\nabla\varphi(x,t))^2+m^2\varphi^2(x,t)\right)$$

Nöther theorem: If $\varphi\to\varphi+\varepsilon\delta\varphi$, $\delta L=\partial_\mu\varepsilon\, g^\mu(\varphi,\delta\varphi)$, then, using equations of motion.

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\varphi}\varepsilon\delta\varphi+\frac{\partial L}{\partial q_{,\mu}}\varepsilon\,\delta\varphi_{,\mu}=\frac{\partial\mathcal{L}}{\partial\varphi}\varepsilon\,\delta\varphi-\partial_\mu\frac{\partial\mathcal{L}}{\partial\varphi}\varepsilon\delta\varphi+\partial_\mu\!\left(\varepsilon\frac{\partial L}{\partial q_{,\mu}}\delta\varphi\right)$$

$\Rightarrow$ Current conservation:

$$\partial_\mu j^\mu=0$$

$$j^\mu=\frac{\partial L}{\partial\varphi_{,\mu}}\delta\varphi-g^\mu(\varphi,\partial\varphi)$$

$\Rightarrow$ Conserved charge

$$\frac{\mathrm{d}}{\mathrm{d}t}Q=0,\quad Q=\int \mathrm{d}^3x\!\left(\frac{\partial\mathcal{L}}{\partial\varphi_{,0}}\delta\varphi-g^0\right)$$

Space translation — energy-momentum conservation: $x^\nu\to x^\nu+\varepsilon^\nu$, $\delta\varphi=\varepsilon^\nu\partial_\nu\varphi$, $\delta\mathcal{L}=\partial_\nu(\varepsilon^\nu\mathcal{L})$.

$$\Rightarrow j^\mu=\left(\frac{\partial L}{\partial\varphi_{,\mu}}\varepsilon^\nu\,\partial_\nu\varphi-\varepsilon^\mu\,\mathcal{L}\right)=\varepsilon^\nu\!\left(\frac{\partial L}{\partial\varphi_{,\mu}}\partial_\nu\varphi-\delta^\mu_\nu\mathcal{L}\right)=\varepsilon^\nu\, j^\mu{}_\nu$$

$j^\mu{}_\nu=$ energy momentum tensor. Equations $\partial_\mu j^\mu{}_\nu=0\Rightarrow$ conservation laws

$$\dot{E}=0,\quad E=\int \mathrm{d}^3x\, j^0{}_0=\int \mathrm{d}^3x(\pi\,\dot{\varphi}-\mathcal{L})$$

$$\dot{\boldsymbol{p}}=0,\quad p_k=\int \mathrm{d}^3x\, j^0{}_k=\int \mathrm{d}^3x(\pi\,\partial_k\varphi)$$

Conversely, momentum generates infinitesimal space translation.

$$[\varphi(x,t),p]_{\mathrm{PB}}=\left[\frac{\partial\varphi}{\partial\varphi(x,t)}\frac{\partial p}{\partial\Pi(x,t)}\right]_{\mathrm{PB}}=\partial_k\varphi(x,t)$$