2008–09–22

Relativistic mechanics

Heart of special relativity: spacetime is different. A change of inertial systems is *not* Galilean transformations:

$$\begin{cases} t' &= t\\ x' &= x - v\,t\\ y' &= y\\ z' &= z\end{cases}$$

but Lorentz transformation

$$t' = \frac{t - x\,v/c^2}{\sqrt{1 - v^2/c^2}}, \quad x' = \frac{x - v\,t}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z.$$

Physical laws are invariant under Lorentz tansformations, not Gallilean transformations. Lorentz transformations resemble rotations invariant distance

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

between two events (t, x, y, z) and $(t + \mathrm{d}t, x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$. Notation: $c\,t = x^0$.

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

Sum μ, ν over $0, 1, 2, 3$.

Making theory relativistic = choosing equations covariant. Vector equations, tensor equations.

Action principle suited to this: Choose the action as a scalar quantity. Build the action from scalar pieces.

For a particle: $A = A_{\text{free}} + A_{\text{int}}$.

Nonrelativistically:

$$A_{\text{free}} = \int_{t_1}^{t_2} \mathrm{d}t\,\frac{1}{2}\,m\,\dot{\boldsymbol{x}}^2 = \frac{1}{2}\,m\,\frac{(\boldsymbol{x}_2 - \boldsymbol{x}_1)^2}{t_2 - t_1}$$

It is not Lorentz invariant. But OK is

$$A_{\text{kin}} = \kappa \int_{t_1}^{t_2} \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \kappa \int \sqrt{c^2 - \dot{\boldsymbol{x}}^2}\,\mathrm{d}t$$

$$L = \kappa\sqrt{c^2 - \dot{\boldsymbol{x}}^2} = \kappa\,c\left(1 - \frac{1}{2}\,\frac{\dot{\boldsymbol{x}}^2}{c^2} + \mathcal{O}\left(\frac{\dot{\boldsymbol{x}}^4}{2}\right)\right) = \kappa\,c - \frac{1}{2}\,\frac{\kappa}{c}\,\dot{\boldsymbol{x}}^2$$

Correct nonrelativistic limit requires $-\frac{1}{2}\cdot\frac{\kappa}{c}\,\dot{\boldsymbol{x}}^2 = \frac{1}{2}\,m\,\dot{\boldsymbol{x}}^2$, so $\kappa = -\,m\,c$.

$$S_{\text{free}} = -\,m\,c\int\,\mathrm{d}s$$

$A_{\rm int}$? Interaction with the electro-magnetic field was described nonrelativistically.

$$A_{\rm EM} = q \int \mathrm{d}t \Big(-\varphi(x,t) + \frac{1}{c} \boldsymbol{A}(\boldsymbol{x},t) \cdot \dot{\boldsymbol{x}} \Big) =$$

(in Gaussian units).

$$= \frac{q}{c} \int \big(\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} - \varphi\, \mathrm{d}x^0 \big) = \frac{q}{c} \int \sum_{\nu=0}^{3} A_\nu\, \mathrm{d}x^\nu$$

if $A_0 = -\varphi$: Study of electrodynamics (Mxwell's equations) shows that A_ν transforms in such a way that $A_{\rm EM}$ is already a Lorentz scalar. A_ν and $\mathrm{d}x^\nu$ are Lorentz vectors, so that $A_{\rm EM}$ is already relativistic.

No big surprise: Special relativity was discovered in electrodynamics.

Summary: Relativistic action for a charged particle in an electromagnetic field is

$$A_{\rm kin} + A_{\rm EM} = \int \Big(-m\,c\, \mathrm{d}s + \frac{q}{c} A_\mu\, \mathrm{d}x^\mu \Big) = \int \Big(-m\,c \sqrt{c^2 - \dot{\boldsymbol{x}}^2} + \frac{q}{c} (\boldsymbol{A} \cdot \dot{\boldsymbol{x}} - c\,\varphi) \Big)\, \mathrm{d}t$$

Relativistic equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \bigg(\frac{m\,c\,\dot{x}^2}{\sqrt{c^2 - \dot{\boldsymbol{x}}^2}} \bigg) = q \bigg(-\nabla\varphi - \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} \nabla \boldsymbol{A} \cdot \dot{\boldsymbol{x}} \bigg) = q \bigg(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \bigg)$$

Other possible types of interactions for a particle.

1) Scalar field. $\chi(\boldsymbol{x},t)$.

$$A_{\rm int} = \kappa \int \chi\, \mathrm{d}s$$

$$A_{\rm free} + A_{\rm int} = \int (1 + \kappa\chi) \sqrt{c^2 - \dot{x}^2}\, \mathrm{d}t$$

Scalar gravity, proposed by Gunnar Nordström 1913. One year before Einstein's general relativity. It is relativistic, but it does not fulfill Einstein's principle of equivalence, which says that in a small region of the universe, gravitational forces can be transformed way by going to an accelerated reference system. Gravity = acceleration.

2) Vector field: Electrodynamics.

3) Tensor field. $T_{\mu\nu}$. Symmetric tensor. Einstein gravity $\eta_{\mu\nu} \to g_{\mu\nu}$. $g_{\mu\nu}$ = metric describing curved spacetime.

$$\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu\, \mathrm{d}x^\nu \quad \to \quad \mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$$

$$-m\,c \int \mathrm{d}s \quad \to \quad -m\,c \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, \mathrm{d}\tau, \quad \dot{x}^\nu \equiv \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}.$$

τ arbitrary curve parameter.

Equations $\Rightarrow$ particle trajectory = geodesic in curved spacetime. Einstein gravity.

$$A_{\rm free} + A_{\rm int} = -m\,c \int \sqrt{\cdots}\, \mathrm{d}\tau$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

On calculation with relativistic actions. It is natural to change from t to general curve parameter τ so that time and space enter symmetrically, as in Jacobi's principle.

Example:

$$A = \int (-m\,c\,\mathrm{d}s + A_\mu\,\mathrm{d}x^\mu) = \int \left(-m\,c\,\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + \frac{q}{c}\,A_\mu\dot{x}^\mu\right)\mathrm{d}\tau$$

with

$$\dot{x}^\mu \equiv \frac{\partial x^\mu}{\partial \tau}, \quad \mu = 0, \ldots, 3$$

Looks like four degrees of freedom, but they are still three. The four degrees of freedom are not independent. (Variation of all includes the parametrisation.)

How to find the Hamiltonian formulation? Usual steps:

1) L as above.

2) $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -m\,c\,\frac{\eta_{\mu\nu}\dot{x}^\nu}{\sqrt{\cdots}} + \frac{1}{c}A_\mu$.

3) $H = p_\mu\,\dot{x}^\mu - L = 0$.

4) From (2), solve for $\dot{x}^\mu = \dot{x}^\mu(p_\mu)$. But impossible: because of an $\dot{x}$ independent relation between the four equations:

$$\left(p_\mu - \frac{q}{c}\,A_\mu\right)\eta^{\mu\nu}\left(p_\nu - \frac{q}{c}\,A_\nu\right) - m^2\,c^2 \equiv H(p,q) = 0$$

($\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$, $\sum_\rho \eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu$.)

What to do? Go back to the definition of the Hamiltonian by Legendre transformations.

$$H(p,x,\dot{x}) = p_\mu\,\dot{x}^\mu - L(x,\dot{x})$$

$$\delta H = \delta p_\mu\,\dot{x}^\mu - \delta x^\mu p_\mu + (\text{eqns})$$

then we concluded from this Hamilton's equations, because x and p could be varied independently. Now we have a constraint $f(p,\,x) = 0$ forbidding independent variations; impossible because of constraint $f(x,p) = 0$. Remedy: Lagrange multipliers.

Result

$$H = p_\mu\,\dot{x}^\mu - L + \frac{\lambda}{2}\,f(p,x) = \frac{1}{2}\,\lambda\left[\left(p_\mu - \frac{q}{c}A_\mu\right)\eta^{\mu\nu}\left(p_\nu - \frac{q}{c}A_\nu\right) - m^2c^2\right]$$

Equations:

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}, \quad \dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = \lambda\,\eta^{\mu\nu}\left(p_\nu - \frac{q}{c}\,A_\nu\right)$$

Interpretation of Lagrange multiplier.

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = \eta_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu = \lambda^2\left(p_\mu - \frac{q}{c}\,A_\mu\right)\eta^{\mu\nu}\left(p_\nu - \frac{q}{c}A_\nu\right) = \lambda^2\,m^2\,c^2$$

$\Rightarrow \lambda$ determines parametrisation τ.

For example, if we want $\tau =$ proper time, $\mathrm{d}s/\mathrm{d}\tau = c$. Corresponds to $\lambda = 1/m$. Then

$$H = \frac{1}{2m}\left(p_\mu - \frac{q}{c}\,A_\mu\right)\eta^{\mu\nu}\left(p_\nu - \frac{q}{c}A_\nu\right) - \frac{1}{2}\,m^2\,c^2$$

The phenomenon that manifest symmetry requires more variables than degrees of freedom, and leads to invariance under transformations depending on continuous parameters (here reparametrisation), is common in relativistic theories. Such reparametrisations are called gauge transformations. One talks about gauge symmetry, gauge invariance, gauge theories.

Another example is electrodynamics, gauge transformation: $\boldsymbol{A} \rightarrow \boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda(x, t)$; $\phi \rightarrow \phi' = \phi + \frac{1}{c}\frac{\partial}{\partial t}\Lambda(x, t)$.

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$