2008–09–16

[This time only Swedish people turned up, so the lecture was, initialy, held in Swedish.]

Hamiltons ekvationer

Ej bättre för att lösa ingenjörsproblem. Better for understanding general properties of mechanics.

Goal: Put time derivative terms in the action in a simple form.

In L: $T = \frac{1}{2}\dot{q}^\nu T_{\mu\nu}(q)\dot{q}^\nu \longrightarrow \frac{1}{2}\dot{q}^\mu\delta_{\mu\nu}\dot{q}^\nu$.

Better option: linearize the action in time derivatives.

$$A_1 = \int \mathrm{d}t\, L(q,\dot{q}) \tag{1}$$

Introduce the new variable $v^\mu = \dot{q}^\nu$. $\dot{q}^\nu - v^\mu = 0$ is implemented by Lagrange multiplier p_μ:

$$\simeq A_2 = \int \mathrm{d}t\, (L(q,\dot{q}) + p_\nu(\dot{q}^\nu - v^\nu)) \tag{2}$$

$$\simeq A_3 = \int \mathrm{d}t\, (p_\nu\dot{q}^\nu - (p_\nu v^\nu - L(q,v))) \tag{3}$$

$$\frac{\delta A_3}{\delta v} = 0: \quad p_v - \frac{\partial L}{\partial v^\nu} = 0$$

solve for v and insert back in to the action.

$$\simeq \quad A_4 = \int \mathrm{d}t\, (p_\nu\dot{q}^\nu - H(p,q))$$

$$H(p,q) = (p_\nu v^\nu - L(q,v))|_{v=v(p,q)}$$

(2)

$$\delta q: \quad \frac{\partial L}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\nu} - \dot{p}_\nu = 0$$

$$\delta p: \quad \dot{q}^\nu - v^\nu = 0$$

$$\delta v: \quad p_\nu = 0$$

(3)

$$\delta q: \quad \frac{\partial L}{\partial q^\nu} - \dot{p}_\nu = 0$$

$$\delta p: \quad \dot{q}^\nu - v^\nu = 0$$

$$\delta v: \quad \frac{\partial L}{\partial v^\nu} - p_\nu = 0, \quad p_\nu \text{ has increased by } \frac{\partial L}{\partial v^\nu}$$

Summary: How to pass from L to H.

1. Define conjugate momenta:

$$p_\nu = \frac{\partial L}{\partial \dot{q}^\nu}$$

2. Solve this for $\dot{q}^\nu = \dot{q}^\nu(p, q)$.

3. Form the Hamiltonian function:

$$H(p, q) = (p_\nu \dot{q}^\nu - L(q, \dot{q})) \quad \Big|_{\dot{q}^\nu = \dot{q}^\nu(p, q)}$$

Action in Hamiltonian formulation:

$$A = \int \mathrm{d}t\, (p_\nu \dot{q}^\nu - H(p, q))$$

Equations of motion: Hamilton's equations.

$$\begin{cases} \dot{p}_\nu &= -\frac{\partial H}{\partial q^\nu} \\ \dot{q}^\nu &= \frac{\partial H}{\partial p^\nu} \end{cases}$$

Example of determining $H(p, q)$:

$$L(q, \dot{q}) = \frac{1}{2}\, \dot{q}^\mu\, T_{\mu\nu}(q)\, \dot{q}^\nu - V(q)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\nu} = T_{\mu\nu}(q) \dot{q}^\nu$$

In matrix notation $\boldsymbol{p} = \mathbb{T} \dot{\boldsymbol{q}}$.

$$\dot{\boldsymbol{q}} = \mathbb{T}^{-1} \boldsymbol{p} \quad \text{or} \quad \dot{q}^\mu = (T^{-1})^{\mu\nu} p_\nu$$

$$H = [= p\, \dot{q} - L = p_\nu \dot{q}^\nu - \frac{1}{2} p_\nu \dot{q}^\nu + V =] = \frac{1}{2} p_\mu (T^{-1})^{\mu\nu} p_\nu + V(q)$$

More common way to derive Halmilton's equations:

Legendre transformation

Form $H(q, \dot{q}, p) \equiv p\, \dot{q} - L(q, \dot{q})$.

$$\delta H = \delta p\, \dot{q} + \delta \dot{q} \underbrace{\left(p - \frac{\partial L}{\partial \dot{q}}\right)}_{=0 \text{ by def. of } p} - \delta q \frac{\partial L}{\partial q} = \left[\frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} = \dot{p}\right] = \delta p\, \dot{q} - \delta q\, \dot{p}$$

$$\Rightarrow \begin{cases} \dot{q} &= \frac{\partial H}{\partial p}\Big|_q \\ \dot{p} &= -\frac{\partial H}{\partial q}\Big|_p \end{cases}$$

$$\begin{cases} \dot{p}_\nu &= -\frac{\partial H}{\partial q^\nu} \\ \dot{q}^\nu &= \frac{\partial H}{\partial p_\nu} \end{cases}$$

Remark: cyclic variables are easier to eliminate in the Hamiltonian formulation. Assume q^1 is cyclic: $H = H(p_1, \ldots, p_n; q^2, \ldots, q^n)$

$$\Rightarrow \quad \dot{p}_1 = -\frac{\partial H}{\partial q'} = 0, \quad p_1 = \text{constant}$$

$H(\text{constant}, p_2, \ldots, p_n; q^2, \ldots, q^n)$. Modification of L has already been performed when going to H.

Remark: In the Lagrangian formulation the position of the system is given by coordinates q^ν. One regards the system as a point moving in configuration space. (The configuration space is the n dimensional manifold on which the system moves; q^ν are the coordinates on this manifold.) To describe the state of a system, you must give all coordinates and velocities. $2n$ real numbers. Here n is the number of degrees of freedom of the system.

In the Hamiltonian formulation the system is a point in phase space. (The phase space is a $2n$ dimensional space with coordinates p_ν and q^ν.) The position in phase space determines the state of the system. Hamilton's equations of motion then determine uniquely the future motion. The equations determine a *flow in phase space*.

Liouville's theorem: "The phase fluid is incompressible." If one picks a region Ω of phase space, it moves in such a way that its volume is time independent.

Proof: Consider first an incompressible fluid in three dimensions.

$$V = \int_\Omega \mathrm{d}^3 x =$$

$$\mathrm{d}V = V(t + \mathrm{d}t) - V(t) = \int_{\Omega(t+\mathrm{d}t)} \mathrm{d}^3 x - \int_\Omega \mathrm{d}^3 x = \int_{\partial\Omega} \mathrm{d}t\, \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = [\text{Gauss}] = \mathrm{d}t \int_\Omega \mathrm{d}V\, (\nabla \cdot \boldsymbol{v})$$

Conclusion: Incompressible fluid $\Leftrightarrow \nabla \cdot \boldsymbol{v} = 0$.

In phase space:

$$\boldsymbol{v} = (\dot{p}_1, \ldots, \dot{p}_n; \dot{q}^1, \ldots, \dot{q}^n) = \left(-\frac{\partial H}{\partial q^1}, \ldots, -\frac{\partial H}{\partial q^n}; \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n} \right)$$

$$\nabla \cdot \boldsymbol{v} = \frac{\partial \dot{p}_1}{\partial p_1} + \cdots = -\frac{\partial^2 H}{\partial p_1 \partial q^1} - \cdots - \frac{\partial^2 H}{\partial p_n \partial q^n} + \frac{\partial^2 H}{\partial q^1 \partial p_1} + \cdots + \frac{\partial^2 H}{\partial q^n \partial p_n} = 0$$

Remark: Liouville's theorem is one of many. There is one integral invariant for each number $1, \ldots, 2n$.

Note: Phase space (coordinates p, q) has dimension $2n$. There is also an *extended* phase space, with dimension $2n + 1$ (t is added as an extra coordinate) and an extended phase space of dimension $2n + 2$ to which t and p_t are added.