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Today: Connection symmetry—conservation law; Jacobi's and Fermat's principles.

(A) For each conserved quantity (function of q, $\dot{q}$ that is constant in time) there corresponds (B) a continuous symmetry. (Action invariance under transformation.), $A \Longleftrightarrow B$.

Nöther's theorem: is a method for $B \Rightarrow A$ in Lagrangian mechanics.

Infinitesimal transformation $q \mapsto q' = q + \delta q$, $\delta q^{\nu} = \varepsilon\, f^{\nu}(q, \dot{q})$.

$$\delta L = L(q', \dot{q}') - L(q, \dot{q}) = \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\, g(q, \dot{q}))$$

$$\delta L = \frac{\partial L}{\partial q^{\nu}}\, \delta q^{\nu} + \frac{\partial L}{\partial \dot{q}^{\nu}}\, \delta \dot{q}^{\nu}$$

I use the summation convention, where any greek index appears twice, once upstairs, once downstairs, is summed over.

$$\text{For example: } \frac{\partial L}{\partial q^{\nu}}\, \delta q^{\nu} \equiv \sum_{\nu} \frac{\partial L}{\partial q^{\nu}}\, \delta q^{\nu}$$

$$\delta L = \frac{\partial L}{\partial q^{\nu}}\, \delta q^{\nu} + \frac{\partial L}{\partial \dot{q}^{\nu}}\, \delta \dot{q}^{\nu} = \underbrace{\left(\frac{\partial L}{\partial q^{\nu}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^{\nu}}\right)}_{=0} \delta q^{\nu} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^{\nu}}\, \delta q^{\nu}\right)$$

$$Q = \frac{\partial L}{\partial \dot{q}^{\nu}}\, f^{\nu}(q, \dot{q}) - g(q, \dot{q})$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(q, \dot{q}) = 0$$

Three standard examples:

1. Energy — time translation

$$q'^{\,\nu}(t) = q^{\nu}(t + \varepsilon) = q^{\nu} + \varepsilon \dot{q}^{\nu}$$

$$\delta q^{\nu} = \varepsilon \dot{q}^{\nu}$$

$$g = L(q, \dot{q})$$

$$Q = \frac{\partial L}{\partial \dot{q}^{\nu}} - L = \text{Energy, normally}$$

$$\dot{Q} = 0 \text{ if } L \text{ does not depend explicitly on time.}$$

Energy is conserved if L does not depend explicitly on time.

2. Spatial translation — momentum conservation.

Go back to Cartesian coordinates for the particles + Lagrange multipliers for constraints. Assume Lagrange multipliers are not transformed:

$$\delta \boldsymbol{x}_i = \varepsilon\, \boldsymbol{n}$$

$$\delta L = 0, \quad \delta \lambda = 0$$

$$Q = \sum_i \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i} \boldsymbol{n} = \sum_i \boldsymbol{p}_i \cdot \boldsymbol{n} = \boldsymbol{P} \cdot \boldsymbol{n}$$

Symmetry of $L \Rightarrow$ momentum in direction $\boldsymbol{n}$ is constant in time.

3. Rotation — angular momentum.

$$\delta \boldsymbol{x}_i = \varepsilon\, \boldsymbol{n} \times \boldsymbol{x}_i, \quad \delta\lambda = 0, \quad \delta L = 0$$

$$Q = \sum_i \frac{\partial L}{\partial \dot{\boldsymbol{x}}_i}(\boldsymbol{n} \times \boldsymbol{x}_i) = \sum_i \boldsymbol{p}_i \cdot (\boldsymbol{n} \times \boldsymbol{x}_i) =$$

$$= \sum_i \boldsymbol{n} \cdot (\boldsymbol{x}_i \times \boldsymbol{p}_i) = \sum_i \boldsymbol{n} \cdot \boldsymbol{l}_i = \boldsymbol{n} \cdot \boldsymbol{L}$$

Invariance of $L \Rightarrow$ Conservation of angular momentum.

Another special case:

Suppose L is independent of q^1 but not $\dot{q}^1$. $\Rightarrow$Symmetry $\delta q' = \varepsilon, \delta L = 0, \delta q^\nu = 0 \,\text{for}\, \nu > 1$.

$$\Rightarrow Q = \frac{\partial L}{\partial \dot{q}^1} = p_1 \text{ is conserved.}$$

Such variables can be called cyclic variables (sometimes ignorable variables).

Is it then possible to eliminate q^1 and $\dot{q}^1$ from L?

Change of notation: $L(q^2, \ldots, q^n; \dot{q}^1, \ldots, \dot{q}^n) = L(q, \dot{q}, \dot{Q})$, where $q, \dot{q}$ are the remaining q's.

$$\frac{\partial L(q, \dot{q}, \dot{Q})}{\partial \dot{Q}} = p = \text{constant in time}$$

Solve this equation for $\dot{Q} = \dot{Q}(q, \dot{q}, p)$. Insert this in L:

$$\tilde{L}(q, \dot{q}, p) \equiv L(q, \dot{q}, \dot{Q}(q, \dot{q}, p))$$

$\tilde{L}$ does not work as a Lagrangian for $q, \dot{q}$:

$$\frac{\partial \tilde{L}}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^\nu} = \underbrace{\frac{\partial L}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\nu}}_{=0\,\text{by equations}} + \underbrace{\frac{\partial L}{\partial \dot{Q}}}_{=P}\frac{\partial \dot{Q}}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{\partial L}{\partial \dot{Q}}}_{=P}\frac{\partial \dot{Q}}{\partial \dot{q}^\nu} = P\left(\frac{\partial \dot{Q}}{\partial q^\nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{Q}}{\partial \dot{q}^\nu}\right) \neq 0$$

Remedy:

$$L^{\text{mod}}(q, \dot{q}, p) \equiv L(q, \dot{q}, Q(q, \dot{q}, p)) - P\dot{Q}(q, \dot{q}, p)$$

Example: Planetary orbits:

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - V(r)$$

θ is cyclic.

$$\Rightarrow p_\theta = \frac{\partial L}{\partial \dot{Q}} = m r^2 \dot{\theta} = \text{angular momentum=conserved}$$

$$\Rightarrow L^{\text{mod}} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{p_\theta^2}{m r^2} - V(r) - \frac{p_\theta^2}{m r^2} = \frac{1}{2} m \dot{r}^2 - \frac{1}{2}\frac{p_\theta^2}{m r^2} - V(r)$$

The new term is the potential for the centrifugal force.

Jacobi's principle in mechanics corresponds to Fermats principle in optics.

Until now the integration variable was time. Let's change the integration variable to a curve parameter τ such that $t(\tau_1) = t_1$, $t(\tau_2) = t_2$. Notation:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = t', \quad \frac{\mathrm{d}q}{\mathrm{d}\tau} = q' = \dot{q}\, t', \quad q(\tau) = q(t(\tau))$$

$$A = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t = \int_{\tau_1}^{\tau_2} L\left(q, \frac{q'}{t'}, t\right) t' \mathrm{d}\tau$$

Now $t(\tau)$ can be treated in the same way as the q^ν's, it is a new generalised coordinate. Action must be stationary under variations of t. Now, suppose $L(q,\, \dot{q},\, t) = L(q,\, \dot{q})$ does not depend explicitly on t. Then t in the new formulation is cyclic. Elimination of t then leads to Jacobi's principle.

More explicitly, assume

$$L = \frac{1}{2}\, \dot{q}^\mu T_{\mu\nu}(q)\, \dot{q}^\nu - V(q)$$

$$L(q, q', t') = \frac{1}{2} \frac{q'^\mu T_{\mu\nu} q'^\nu}{t'} - t'\, V(q)$$

$$p_t = \frac{\partial L}{\partial t'} = -\frac{1}{2} \frac{q'^\mu T_{\mu\nu} q'^\nu}{t'^2} - V(q)$$

$=-E$.

Solving for t':

$$t' = \sqrt{\frac{q'^\mu T_{\mu\nu} q'^\nu}{2(E - V)}}$$

$$L^{\mathrm{mod}}(q, q') = L - t'\, p_t = \frac{1}{2} \frac{q'^\mu T_{\mu\nu} q'^\nu}{t'} - t'\, V - t'\left(-\frac{1}{2} \frac{q'^\mu\, T_{\mu\nu} q'^\nu}{t'^2} - V\right) = \frac{q'^\mu\, T_{\mu\nu}\, q'^\nu}{t'} =$$

$$= \sqrt{2(E - V) q'^\mu T_{\mu\nu} q'^\nu}$$

$$\text{Jacobi's principle:} \quad A^{\mathrm{mod}} = \int \mathrm{d}\tau \sqrt{2(E - V)\, q'^\mu T_{\mu\nu} q'^\nu}$$

Note: Resembles Fremat's principle in optics.

"In a medium with refractive index $n(\boldsymbol{r})$, light rays between 2 points choose the way that minimizes the optical distance:

$$\int_{r_1}^{r_2} n(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r} = \int_{s_1}^{s_2} n(\boldsymbol{r}(s)) \sqrt{\left(\frac{\mathrm{d}\boldsymbol{r}(s)}{\mathrm{d}s}\right)^2}\, \mathrm{d}s$$

Fermat's principle is explained by the wave nature of light, and the fact that wave length $\propto 1/n$.

Is there a similar explanation of Jacobi's principle? Yes, particles have wave nature according to quantum mechanics.